

More Theorem-Proving

The primary purpose of these notes is to supplement the textbook. We are covering the material roughly in order and in about the same way. But the way that I explain some topics is different than the author's way, and I figure that hearing a new concept from a couple of different perspectives is a good thing.

Having said that, we have reached a couple of sections in which there is very little in the way of "new" material and the author's do a great job of explaining theorem-proving. I have relatively little to add. I therefore want to HIGHLY encourage you to read Sections 1.5 and 1.6 in our text. In particular, the Addendum on pages 36-38 is highly informative.

Having said THAT, I do want to emphasize some of their points and provide a couple more examples. As I mentioned in the last set of notes, proving something by contradiction is a very powerful method. It allows you to assume more information, and many times that's handy. But when should you use it? Of course there is no hard and fast rule "Always use Proof By Contradiction when..." But if you are asked to show that something is *not* true (or show a certain property does *not* hold), you can often assume that it is true (or does hold) and find a contradiction.

Example 1: Prove that there not exist three consecutive natural numbers such that the cube of the largest is equal to the sum of the cubes of the other two.

Proof: Since this problem asks us to prove that something does NOT exist, I'll assume it does and attempt to arrive at a contradiction. So suppose there exist three consecutive natural numbers such that the cube of the largest is equal to the sum of the cubes of the other two. Let's call the smallest one n and then the other two would be $n + 1$ and $n + 2$. So we are assuming that $n^3 + (n + 1)^3 = (n + 2)^3$. If we work out the algebra, we get:

$$\begin{aligned}n^3 + n^3 + 3n^2 + 3n + 1 &= n^3 + 6n^2 + 12n + 8 \\n^3 - 3n^2 - 9n - 9 &= 0 & (*) \\n^3 &= 3n^2 + 9n + 9\end{aligned}$$

$$n^3 = 3(n^2 + 3n + 3)$$

Since 3 divides the right-hand side, it must also divide the left. Since 3 is prime and it divides n^3 , it must also divide n . So that means that $n = 3k$ for some natural number k . Substituting this into (*) we have,

$$\begin{aligned}(3k)^3 - 3(3k)^2 - 9(3k) - 9 &= 0 \\ 27k^3 - 27k^2 - 27k - 9 &= 0 \\ 27(k^3 - k^2 - k) &= 9\end{aligned}$$

Reasoning as we just did, if 27 divides the left-hand side, it must divide the right-hand side. † [27 does not divide 9.] Therefore, there does not exist three consecutive natural numbers such that the cube of the largest is equal to the sum of the cubes of the other two.

We have discussed several different theorem-proving methods: direct, contrapositive, and contradiction. However, even within each of these methods are various techniques.

Example 2: If x is a real number for which $x^2 + 5x + 6 < 0$, then $-3 < x < -2$.

Proof: In this proof, we'll do what is called "element-chasing." We'll choose an arbitrary element that satisfies the hypothesis, and check that the conclusion holds too. So suppose that x is a real number for which $x^2 + 5x + 6 < 0$. Since $x^2 + 5x + 6 = (x + 2)(x + 3)$ and since the product of two numbers is negative if and only if one of the two numbers is positive and the other is negative, we have two cases to consider: (1) $x + 2 > 0$ and $x + 3 < 0$, and (2) $x + 2 < 0$ and $x + 3 > 0$.

Case (1): In this case, we have that $x > -2$ and $x < -3$. But this is impossible. No real number has this property. So case (1) cannot occur.

Case (2): In this case, we have $x < -2$ and $x > -3$; that is $-3 < x < -2$.

Two comments are in order. First, you need to be careful to distinguish between the valid theorem-proving strategy "element-chasing" and the invalid

strategy “proof by example.” It was important that the element we picked was arbitrary. You cannot prove a property holds for all real numbers (or natural numbers, or whatever) by simply selecting one number that has the property. Secondly, we actually used another technique during this last proof: “proof by cases.” Many times you will want to break up the problem into cases because they will be easier to handle independently of each other. Here is another example of this method.

Example 3: Prove that if n is a natural number, then $n^2 + n$ is even.

Proof: Case (1): If n is even, then it can be written as $n = 2k$ for some natural number k . Therefore,

$$n^2 + n = (2k)^2 + (2k) = 4k^2 + 2k = 2(2k^2 + k),$$

which is clearly even.

Case (2): If n is odd, then it can be written as $n = 2k - 1$ for some natural number k . Therefore,

$$n^2 + n = (2k - 1)^2 + (2k - 1) = 4k^2 + 6k - 2 = 2(2k^2 + 3k - 1),$$

which is clearly even.